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# On the rôle of NLL corrections and Energy Conservation in the High Energy Evolution of QCD

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**ABSTRACT:** We present a new method for solving the BFKL evolution applicable at both leading and next-to-leading logarithmic accuracy, and tailored to the study of QCD multi-jet events at colliders. We utilise this to discuss corrections to the standard analysis. There are known, large corrections from energy and momentum conservation. We show that, despite claims to the contrary in the literature, these are unrelated to the next-to-leading logarithmic corrections to the evolution kernel.

**KEYWORDS:** .

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## 1. Introduction

One of the many immediate challenges for QCD is to provide a reliable description of the multiple hard jet environment of the LHC. Besides posing a very interesting problem in itself, the QCD dynamics will provide signals similar to that of many sources of physics beyond the standard model, and so is very important to understand in detail. An intriguing alternative to the standard approach of calculating the production rate of a few hard partons by fixed order perturbation theory is to use the framework arising from the BFKL (Balitskii–Fadin–Kuraev–Lipatov) equation to calculate the emission of gluons (and quarks at next-leading logarithmic accuracy) from the evolution of an effective, Reggeized gluon (Reggeon) propagator. The starting point here is the observation that for e.g.  $2 \rightarrow 2, 2 \rightarrow 3, \dots$  gluon scattering, the partonic cross section in the limit where the rapidity span of the two leading gluons is increased is dominated by the contribution from Feynman diagrams with a  $t$ -channel gluon exchange. This  $t$ -channel gluon is then evolved according to the BFKL equation, and will emit partons accordingly. Starting from the  $2 \rightarrow 2$ -gluon exchange, the  $2 \rightarrow 2+n$  gluon scattering process can be calculated in the limit of large rapidity spans  $\Delta$ , thanks to the Regge factorisation of the colour octet exchange. Obviously, this means that the formalism is relevant only if there is sufficient energy at colliders to have multiple emissions spanning large rapidity intervals. In this high energy limit, the partonic cross section  $(p'_a, p'_b \rightarrow p_a, \{p_i\}, p_b)$  factorises as

$$d\hat{\sigma}(p_a, p_b) = \Gamma_a(\mathbf{p}_a) f(\mathbf{p}_a, -\mathbf{p}_b, \Delta) \Gamma_b(\mathbf{p}_b), \quad (1.1)$$

where  $p_a, p_b$  is the momentum of the partons furthest apart in rapidity,  $\Gamma_{a,b}$  are the process dependent impact factors, describing here the gluon–gluon–Reggeised-gluon–coupling, and  $f(\mathbf{p}_a, -\mathbf{p}_b, \Delta)$  is the process independent gluon Green’s function, which is the object evolving according to the BFKL equation. We use bold face vectors to denote the transverse components. This is the same gluon Green’s function that would enter calculations of the

small- $x$  evolution of parton density functions. The evolution of the gluon Green's function can of course be discussed generally without reference to specific impact factors, and so the conclusions of this study should impact all applications of the BFKL formalism for colour octet or inelastic studies. In Eq. (1.1), the partonic cross section has been integrated over any number of gluons emitted in this evolution, so only the dependence on the momenta of the leading jets have been retained. The BFKL approach should give a good approximation to the full  $2 \rightarrow 2+n$  calculation, when the hard jets are well separated in rapidity.

The BFKL equation governing the evolution of the  $t$ -channel Reggeised gluon was solved iteratively in Ref. [1–3] using the evolution at leading logarithmic accuracy [4–6], and more recently in Ref. [7, 8] at full next-to-leading logarithmic accuracy [9, 10]. The present paper has three purposes. Firstly, to present a new method for obtaining the QCD evolution according to the BFKL equation. Secondly, to discuss the implications of this new formalism on our understanding of the sources of corrections. And thirdly, to announce the availability of a computer code that calculates the exclusive multi-parton production rate expected at the LHC or Tevatron according to the BFKL evolution.

## 2. Iterative Solution of the BFKL Equation

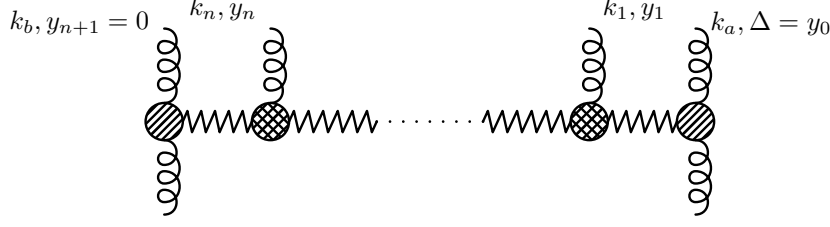
Our starting point for the discussion is the fully inclusive BFKL equation describing the evolution of the gluon Green's function  $f(\mathbf{k}_a, \mathbf{k}_b, \Delta)$

$$\omega f_\omega(\mathbf{k}_a, \mathbf{k}_b) = \delta^{(2+2\epsilon)}(\mathbf{k}_a - \mathbf{k}_b) + \int d^{2+2\epsilon} \mathbf{k} \mathcal{K}_\epsilon(\mathbf{k}_a, \mathbf{k} + \mathbf{k}_a) f_\omega(\mathbf{k} + \mathbf{k}_a, \mathbf{k}_b), \quad (2.1)$$

where  $w$  is the Mellin-conjugated variable to  $\Delta$ , and the BFKL kernel  $\mathcal{K}_\epsilon(\mathbf{k}_i, \mathbf{k}_j)$  is presently known to next-to-leading logarithmic accuracy, where the logarithm is  $\ln(s_{ij}/|\mathbf{k}_i||\mathbf{k}_j|)$ , and  $s_{ij}$  the invariant mass of partons  $i$  and  $j$ . The solution to the evolution from a momentum  $\mathbf{k}_b$  at rapidity  $y_b$  to  $\mathbf{k}_a$  at rapidity  $y_a$  according to this integral equation can be written on the form

$$\begin{aligned} f(\mathbf{k}_a, \mathbf{k}_b, \Delta) &= \sum_{n=0}^{\infty} \int d\mathcal{P}_n \mathcal{F}_n, \\ \int d\mathcal{P}_n &= \left( \int \prod_{i=1}^n d\mathbf{k}_i \int_0^{y_0} dy_1 \int_0^{y_1} dy_2 \cdots \int_0^{y_{n-1}} dy_n \right) \delta^{(2)} \left( \mathbf{k}_a + \sum_{l=1}^n \mathbf{k}_l - \mathbf{k}_b \right) \\ \mathcal{F}_n &= \left( \prod_{i=1}^n e^{\omega(\mathbf{q}_i)(y_{i-1}-y_i)} V(\mathbf{q}_i, \mathbf{q}_{i+1}) \right) e^{\omega(\mathbf{q}_{n+1})(y_n-y_{n+1})} \end{aligned} \quad (2.2)$$

with  $y_0 = y_a \equiv \Delta$ ,  $y_b = y_{n+1} = 0$ ,  $\mathbf{q}_i = \mathbf{k}_a + \sum_{l=1}^{i-1} \mathbf{k}_l$ , and, crucially, the real production vertices  $V(\mathbf{q}_i, \mathbf{q}_{i+1})$  and trajectories  $\omega(\mathbf{q}_i)(y_{i-1} - y_i)$  are regularised and finite at LL [1–3] and NLL [7, 8] in order to facilitate a direct numerical evaluation. The correctness of the procedure at NLL was proved in Ref. [11] by comparing to analytic results for the evolution in  $\mathcal{N}=4$  Super Yang-Mills theory, since the analytic solution to the BFKL equation is only known for conformal invariant theories (please see Ref. [12] for a discussion of the analytic methods at NLL).



**Figure 1:** The  $2 \rightarrow 2+n$  gluon scattering process described using Regge factorisation, with the initial state at the bottom. The shaded blobs are the gluon-gluon-Reggeon impact factors  $\Gamma_{a,b}$ , and the hatched blobs are the (regularised) gluon-Reggeon-Reggeon vertices. The Reggeized gluon (Reggeon) propagators are marked with zigzag lines. Gluon emission is generated in the rapidity span between the impact factors by the evolution described by the BFKL equation of the Reggeized gluon. At NLL the vertices can emit one or two gluons, or a quark-anti-quark pair.

The formulation of the solution to the BFKL evolution in Eq. (2.2) allows in principle for the calculation of the fully exclusive  $2 \rightarrow 2+n$  cross section, i.e. the expression of the differential cross section in Eq. (1.1) as  $d\hat{\sigma}(p_a, \{p_i\}, p_b)$ . It is reassuring that this picture is fully consistent with recent results on the multi-Regge form of QCD amplitudes at the next-to-leading logarithmic level [13–15].

### 3. Direct Solution of the BFKL Evolution

Starting from Eq. (2.2) it is possible to construct a new method of evolving according to the BFKL kernel by first performing a simple change of variables in the nested integration over rapidities, followed by the introduction of another integral and delta functional, in order to make the integration over rapidity separations independent

$$\int_0^{y_0} dy_1 \int_0^{dy_1} dy_2 \cdots \int_0^{dy_{n-1}} dy_n \left( \prod_{i=1}^n e^{\omega(\mathbf{q}_i)(y_{i-1}-y_i)} \right) e^{\omega(\mathbf{q}_{n+1})(y_n-y_{n+1})} \quad (3.1)$$

$$= \int_0^\Delta d\delta y_n \int_0^{\Delta-y_n} d\delta y_{n-1} \cdots \int_0^{\Delta-y_n-\cdots-y_2} d\delta y_1 \left( \prod_{i=1}^n e^{\omega(\mathbf{q}_i)\delta y_i} \right) e^{\omega(\mathbf{q}_{n+1})\delta y_{n+1}} \quad (3.2)$$

$$= \int_0^\infty d\delta y_{n+1} \int_0^\infty d\delta y_n \cdots \int_0^\infty d\delta y_1 \delta\left(\Delta - \sum_{i=1}^{n+1} \delta y_i\right) \prod_{i=1}^{n+1} e^{\omega(\mathbf{q}_i)\delta y_i}, \quad (3.3)$$

so the gluon Green's function  $f(\mathbf{k}_a, \mathbf{k}_b, \Delta)$  in Eq. (2.2) can be written as

$$\begin{aligned} f(\mathbf{k}_a, \mathbf{k}_b, \Delta) &= \sum_{n=0}^{\infty} \int d\mathcal{P}_n \mathcal{F}_n, \\ \int d\mathcal{P}_n &= \left( \prod_{i=1}^n \int d\mathbf{k}_i \int_0^\infty d\delta y_i \right) \int_0^\infty d\delta y_{n+1} \delta^{(2)}\left(\mathbf{k}_a + \sum_{l=1}^n \mathbf{k}_l - \mathbf{k}_b\right) \delta\left(\Delta - \sum_{i=1}^{n+1} \delta y_i\right) \\ \mathcal{F}_n &= \left( \prod_{i=1}^n e^{\omega(\mathbf{q}_i)\delta y_i} V(\mathbf{q}_i, \mathbf{q}_{i+1}) \right) e^{\omega(\mathbf{q}_{n+1})\delta y_{n+1}} \end{aligned} \quad (3.4)$$

Using this, we see that  $f(\mathbf{k}_a, \mathbf{k}_b, \Delta)$  is simply the value at  $\Delta \equiv \sum_{i=1}^{n+1} \delta y_i$  of the phase space integral of the product of real-emission vertices  $V(\mathbf{q}_i, \mathbf{q}_{i+1})$  at rapidity  $y_i = \sum_{j=1}^i \delta y_j$  connected with Regge factors  $e^{\omega(\mathbf{q}_i)\delta y_i}$  describing the probability of no (resolved) emission between two adjacent (in rapidity) vertices<sup>1</sup>. In this case, it is clear that the *virtual and unresolved* corrections encoded in  $\omega(\mathbf{q}_i)$  lead to a *decrease* with increasing  $\Delta$ , while any *increase* in  $f(\mathbf{k}_a, \mathbf{k}_b, \Delta)$  is due to the *integration over phase space of the resolved emission* from the vertices  $V(\mathbf{q}_i, \mathbf{q}_{i+1})$ . The BFKL evolution can then be found by the following algorithm :-

1. Choose a random number of vertices for the evolution,  $n \geq 0$
2. Generate a set  $\{\mathbf{k}_i\}_{i=1,\dots,n}$  of transverse momenta (the outgoing momenta are  $\{-\mathbf{k}_i\}_{i=1,\dots,n}$ )
3. Calculate the corresponding set of trajectories  $\{\omega(\mathbf{q}_i)\}_{i=1,\dots,n+1}$ , and vertex factors  $\{V(\mathbf{q}_i, \mathbf{q}_{i+1})\}_{i=1,\dots,n}$ ,  $\mathbf{q}_i = \mathbf{k}_a + \sum_{l=1}^{i-1} \mathbf{k}_l$
4. Generate the inter-vertex rapidity separations  $\{\delta y_i\}$  according to the distributions  $e^{\omega(\mathbf{q}_i)\delta y_i}$
5. Calculate the corresponding  $\Delta = \sum_{i=1}^{n+1} \delta y_i$  and return  $\prod_{i=1}^n V(\mathbf{q}_i, \mathbf{q}_{i+1})$
6. Repeat until required Monte Carlo accuracy is obtained

This algorithm is vastly superior to the immediate Monte Carlo implementation of the phase space integrals in Eq. (2.2). This is because that instead of calculating, for a given  $\Delta$ , the contribution from any number of emissions and any momentum configuration of these, this new method calculates a representative rapidity span for a given number of emissions and their configurations of momenta. Furthermore, this method is explicitly symmetric in  $k_a \leftrightarrow k_b$  and evolution direction (increasing or decreasing rapidities), and it offers a more direct way of implementing the necessary study at colliders of any length of the rapidity interval  $\Delta$ , since this is automatically achieved by just a single sum over any number of emissions. Unweighting of the Monte Carlo is also significantly more efficient, since part of the integrand in the previous formulation contributing to the variance of the integrand is now used to determine a representative value for  $\Delta$ . We stress that both approaches provide the same correct solution to the BFKL evolution.

#### 4. Phase space restrictions from energy and momentum conservation

Besides leading to a more efficient implementation of the BFKL evolution, the new formulation of the solution to the BFKL evolution is also useful for discussing sources of corrections to the standard BFKL analysis. The phase space restrictions from energy and momentum conservation can be implemented simply by adding a step in the algorithm rejecting events prohibited by phase space considerations. We see that the discussion of the region of integration for real emission is completely independent of the logarithmic

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<sup>1</sup>This interpretation holds for  $\omega(\mathbf{q}_i) < 0$ , which can always be fulfilled in the regularisation procedure of Ref. [1–3] at LL and Ref. [7, 8] at NLL.

accuracy of the evolution kernel expressed in terms of the vertices  $V(\mathbf{q}_i, \mathbf{q}_{i+1})$  and Regge trajectories  $\omega(\mathbf{q}_i)$ . This clearly shows that, despite what is often claimed in the literature (see e.g. Ref. [16] and references therein), phase space restrictions in terms of e.g. energy and momentum conservation are completely independent of the logarithmic accuracy to which the BFKL evolution is performed. Specifically, the NLL corrections to the evolution kernel *do not* implement energy and momentum conservation. Furthermore, the standard solution of the LL and guestimate of the NLL evolution based on a Mellin transform in the transverse momentum of the kernel will *always fail* to take such considerations into account, by its very nature as fully inclusive both in number of emissions and as an integral to infinity of the transverse momentum. This is simply a result of the fact that the BFKL evolution is local, while energy and momentum conservation depends on the full final state configuration. Such considerations should be implemented by modifications to the evolution beyond the discussion of the logarithmic accuracy of the evolution. Please note that this discussion of energy and momentum conservation goes beyond the discussion of longitudinal momentum in collinear splittings at small- $x$ , implemented by ensuring the vanishing of the first moment (see e.g. Ref. [17]).

While it is not our job to guess the cause of the confusion in the literature over the rôle of energy and momentum conservation and higher logarithmic corrections to the evolution kernel, it is perhaps beneficial to discuss the differences and similarities between the multi-particle generating colour-octet exchange and the colour-singlet exchange relevant for diffractive studies. The evolution in rapidity  $\Delta$  of the gluon Green's function for both cases is described by a BFKL equation. In the case of diffractive  $2 \rightarrow 2$  processes, the rapidity span of the BFKL evolution is given by  $\Delta \approx \ln s/s_0$ , when the Regge scale  $s_0 = |\mathbf{k}_a||\mathbf{k}_b|$  is the product of the transverse momentum of the two (massless) scattered particles. However, the impact factors depend on  $s_0$  only at NLL accuracy, and so it can be argued that only at NLL accuracy does the prediction gain a correct dependence on the centre of mass energy (although of course the evolution can (and should!) be discussed without reference to specific impact factors). While this is true, care has to be taken when discussing instead colour-octet exchange with multiple emissions. First of all, there is no one-to-one correspondence between the rapidity span of the evolution and the centre of mass energy, although obviously the centre of mass energy tends to increase with increasing rapidity span. Note specifically that if one sets  $s = s_0 e^\Delta$  with the Regge scale  $s_0 = |\mathbf{k}_a||\mathbf{k}_b|$ , then any additional emission from the BFKL evolution is kinematically excluded. The error<sup>2</sup> in ignoring the contribution from the BFKL emission to the centre of mass energy is significant [18, 19]. Secondly, any constraint on the real emission will significantly lower the expected BFKL signatures, e.g. the expected rise in  $F_2$  at small- $x$ , and the increasing jet azimuthal decorrelation with rapidity at hadron colliders. In fact, the effect of correctly implementing energy and momentum conservation on top of the LL BFKL evolution has a larger impact on the jet azimuthal decorrelation even at LHC energies than the inclusion of the NLL corrections to the evolution kernel (compare e.g. Fig. 10 of Ref. [8] with Fig. 5 of Ref. [3]).

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<sup>2</sup>It is often argued that this correction is logarithmically subleading. However, as we have demonstrated this is not part of the subleading corrections to the evolution kernel.

## 5. Conclusions

We have presented a new method for solving the BFKL evolution very effectively for the study of multi-partonic final states at hadron colliders. We have furthermore demonstrated that the discussion of energy and momentum conservation and phase space constraints of the evolution in general is completely separate to the discussion of the logarithmic accuracy of the evolution kernel, contrary to the claims found in the literature. We have demonstrated how energy and momentum conservation can be implemented.

A program implementing the new algorithm for the BFKL evolution in the case of multi-jet production at Tevatron and LHC energies is available at the URL

<http://www.hep.phy.cam.ac.uk/~andersen/BFKL>

The current version implements energy and momentum conservation, and the BFKL evolution kernel at LL supplemented by the running coupling terms from NLL.

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